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CONVECTION AND FRICTIONAL HEATING IN A CONE AND PLATE SYSTEM

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Abstract—The temperature field resulting from the frictional heating of a fluid confined in the region between a cone and a plate, which are in relative steady rotation, is calculated for prescribed thermal boundary conditions. The convective heat-transfer problem pertaining to the three dimensional velocity field, characteristic of this geometry, is solved as an asymptotic expansion for small Reynolds number and small cone angle. Together with adaptations of previously published results for plane Couette flow of fluids with temperature-dependent transport properties, these solutions provide quantitative measures of the effects of conduction, convection, frictional heating, as well as the temperature sensitivity of the fluid transport properties.

NOMENCLATURE

Br, Brinkman number; $\mu_0 R^2\Omega^2/k_0\bar{T}_0$;
C_p, heat capacity of fluid [J kg^{-1} K^{-1}];
E^2, differential operator defined by equation (6);
h, local heat-transfer coefficient (42) [J m^{-2} s^{-1} K^{-1}];
k, thermal conductivity of fluid [J m^{-1} s^{-1} K^{-1}];
k_0, thermal conductivity of fluid at temperature $\bar{T}_0$ [J m^{-1} s^{-1} K^{-1}];
M, actual torque value for temperature-dependent transport properties in absence of secondary motions [N m];
$\tilde{M}$, actual torque value for isothermal flow including secondary motions (70) [Nm];
M_0, torque value assuming temperature-independent transport properties and absence of secondary flow [Nm];
Nu, local Nusselt number (42);
Pr, Prandtl number: $\mu C_p/k_0$;
r, radial coordinate nondimensionalized with respect to R;
R, cone radius [m];
Re_0, Reynolds number: $R^2\Omega\rho/\mu_0$;
Re, Reynolds number: $Re_0\varepsilon^2$;
T, dimensionless temperature: $(\bar{T}-\bar{T}_0)/\bar{T}_0$;
$\bar{T}_0$, reference temperature [K];
u_r, u_θ, u_ϕ, dimensionless velocity components in the r, θ and ϕ directions, respectively (non-dimensionalized with respect to $R\Omega$);
w, dimensionless tangential velocity: $r\cos\beta u_\phi$ (2);
w_i, coefficient of term of $O(Re^{2i})$ in asymptotic expansion for w (17).

Greek symbols

$\alpha_1, \alpha_2, \ldots$, dimensionless temperature coefficients of thermal conductivity (48);
$\beta_1, \beta_2, \ldots$, dimensionless temperature coefficients of viscosity (49): also $\beta_j = \bar{\beta}j/jl.$;
$\bar{\beta}$, dimensionless coefficient in exponential viscosity–temperature relation (49);
β, spherical coordinate (Fig. 1): $(\pi/2)-\theta$[rad];
β_0, cone angle [rad];
ε, $\sin\beta_0$;
ζ, $\sin\beta/\varepsilon$;
ζ_m, value of ζ at the maximum temperature;
θ, spherical coordinate (Fig. 1) [rad];
κ_0, dimensionless rotational velocity of plate: Ω_0/Ω;
κ_1, dimensionless rotational velocity of cone: Ω_1/Ω;
μ viscosity of fluid [kg m^{-1} s^{-1}];
μ_0, viscosity of fluid at reference temperature [kg m^{-1} s^{-1}];
ρ, density of fluid [kg m^{-3}];
χ, dimensionless stream function: $\psi/Re_0\varepsilon^3$;
χ_i, coefficient of term of $O(Re^{2i})$ in asymptotic expansion for χ (18);
ψ, dimensionless stream function (2);
ϕ, spherical coordinate (Fig. 1);
Φ_V, dissipation function (14);
Ω_0, angular velocity of plate [rad s^{-1}];
Ω_1, angular velocity of cone [rad s^{-1}];
Ω, reference angular speed: $|\Omega_0|+|\Omega_1|$ [rad s^{-1}].

INTRODUCTION

THIS paper is concerned with convective heat transfer in a fluid confined between a cone and plate which are in relative steady motion (Fig. 1). The fluid flow and heat transfer in specialized fluid processing systems have been of interest to the present authors in a

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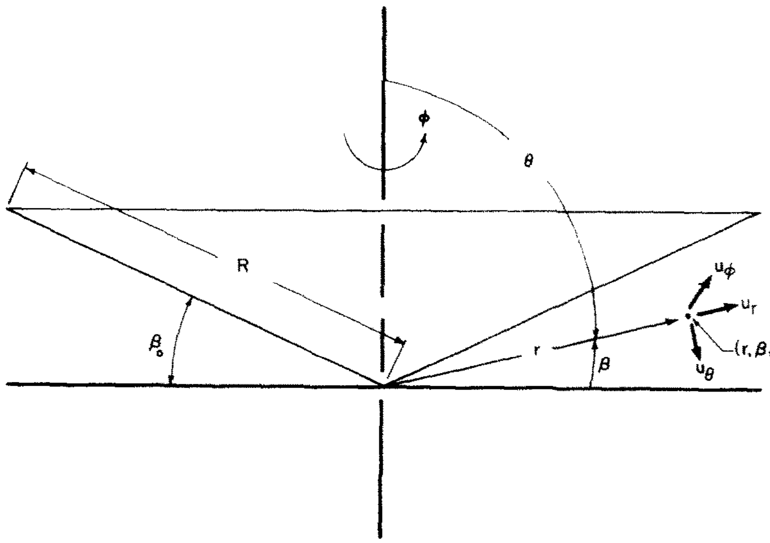

FIG. 1. Cone and plate and coordinate system.

number of previous analytical studies [1, 2, 6, 15, 16]. The present configuration is important in a number of applications, but is most widely used for measurement of rheological properties [1, 2]. Fluid motion in this geometry is inherently three dimensional, being comprised of a primary tangential motion with a superimposed secondary flow. The purely tangential main flow is associated with a limiting condition of uniform stress and strain rate in the fluid region, and this can roughly be surmized from the fact that at any point on the rotating cone surface the velocity is proportional to the distance r from the apex, and, of course, so is the gap width at the same point. Accordingly their ratio, representing a first approximation to the strain rate, is independent of r. This feature, it seems certain, embodies the singular advantage of this geometry over all others used for rheological characterization of shear-sensitive materials. Cone angles, β_0, in general use are small, rarely exceeding 4°, and it turns out this property can be exploited to establish analytical approximations of broad practical utility. The objective of the present work is to examine in some detail the heat-transfer problem associated with the frictional heating of the test fluid in this three-dimensional flow field for various prescribed thermal boundary conditions. To do this, the perturbation solutions of the Navier–Stokes equations developed previously by Turian [1] for this geometry for Newtonian fluids with temperature-independent transport properties are used. It was shown in this earlier work that an appropriately defined Reynolds number, Re^2, and a suitable cone angle function, $\varepsilon^2 = \sin^2 \beta_0$, provide the proper gage functions for ordering the various approximations in the asymptotic expansions for the velocity and pressure fields. These same gage functions are used in the asymptotic analysis presented here to treat the energy equation including a viscous dissipation heat source. The small-Re, small-ε limiting process used here turns out to be equivalent to an expansion about essentially the plane Couette flow approximation as the primary reference state. This then permits the adaptation of previously published results of Turian and Bird [2] for the variable viscosity, variable thermal conductivity analysis of the frictional heating in plane Couette flow to the present situation. As a result, contributions to the heat transfer relating to conduction, convection, and viscous dissipation, as well as those deriving from the temperature sensitivity of the relevant transport properties are depicted in the overall analysis presented here.

There is an extensive literature on frictionally heated flows. A striking phenomenon associated with variable viscosity flows of liquids is the possible occurrence of multiple steady states, and the nonexistence of steady state solutions for so-called stress parameter levels exceeding the critical value. The existence and uniqueness of frictionally heated variable viscosity Newtonian Couette and Poiseuille flows was examined by Joseph [3, 4] for the general fluidity-temperature dependence as well as for the particular cases of the linear, the quadratic and the exponential temperature dependences. Experimental confirmation of the phenomenon for the simple Couette and Poiseuille flows has been reported by Sukanek and Laurance [5] in a paper which contains a comprehensive bibliography on the subject. In frictionally heated variable viscosity liquid flows the critical stress, or the equivalent property, defines the limit at which the frictionally generated heat has reached the full capacity of the system to remove it to the surroundings by all available means (conduction, convection, etc.). Joseph [3, 4] has shown that the critical stress value depends strongly on the temperature sensitivity of the viscosity of the fluid; the more rapidly the viscosity decrease with increasing temperature the lower the value of the critical stress. For pseudoplastic non-Newtonian fluids the viscosity decreases with increasing stress as well as with temperature, and consequently temperature and non-Newtonian effects are reinforcing, whereas for dilatant non-Newtonian fluids these effects are counteracting as shown by Turian [6]. An extensive listing of non-Newtonian frictionally heated flows has been given in the review by Trowbridge and Karran [7].

A common characteristic of virtually the entire body of published work on variable viscosity flows described in the foregoing is the fact that it pertains strictly to one-dimensional velocity fields. This reduces the governing energy equation to substantially the one-dimensional steady state conduction form and furthermore it results in an effective uncoupling of the momentum and energy equations. The critical phenomena discussed here arise in other physical situations including heat generation in conducting solids [8], in buoyant viscous flows [9], and in exothermic reactions [10]. The last problem traces its origin to the brilliant ideas of Frank-Kamenetskii and his theories on thermal explosions [11], which have since spawned a truly massive explosion of papers on multiple steady states and associated stability phenomena in chemically reacting systems. Stability problems relating to frictionally heated simple flows have been considered by Joseph [3, 4], by Sukanek *et al.* [12], and most recently by Ho *et al.* [13]. However, the present asymptotic solutions are, as will be shown later, strictly small Brinkman number approximations pertaining to a range of conditions in which these complications are not relevant.

FORMULATION OF THE PROBLEM

For the velocity field referred to in Fig. 1 we take u_r, u_θ, and u_ϕ to be the velocity components non-dimensionalized with respect to $R\Omega$, in which R is the cone radius and $\Omega = |\Omega_0| + |\Omega_1|$ is the reference angular speed. The angular velocities of the plate and cone are designated by Ω_0 and Ω_1, respectively. Using $\beta = (\pi/2) - \theta$ we define the stream function ψ through

$$u_r = \frac{1}{r^2 \cos\beta}\frac{\partial\psi}{\partial\beta}; \quad u_\theta = \frac{1}{r\cos\beta}\frac{\partial\psi}{\partial r} \tag{1}$$

and we set

$$u_\phi = \frac{w}{r\cos\beta}. \tag{2}$$

Symmetry requires that there be no dependence on ϕ, and consequently the stream function defined in equation (1) satisfies the equation of continuity exactly. Further, using the transformation

$$\zeta = \frac{\sin\beta}{\sin\beta_0} = \frac{\sin\beta}{\varepsilon} \tag{3}$$

it can be shown that the Navier–Stokes equations, upon elimination of the pressure, reduce to the form given by

$$-Re_0\left[\frac{\varepsilon}{r^2}\frac{\partial(\psi, E^2\psi)}{\partial(r,\zeta)} + \frac{2\varepsilon}{r^2}E^2\psi\left(\frac{\varepsilon^2\zeta}{1-\varepsilon^2\zeta^2}\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial\zeta}\right)\right.$$
$$\left. + \frac{2\varepsilon^3}{r^2}w\left(\frac{\varepsilon^2\zeta}{1-\varepsilon^2\zeta^2}\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial\zeta}\right)\right] = E^4\psi \tag{4}$$

$$-Re_0\varepsilon\frac{1}{r^2}\frac{\partial(\psi, w)}{\partial(r,\zeta)} = E^2 w \tag{5}$$

with

$$E^2 = \frac{1}{r^2}\frac{\partial^2}{\partial\zeta^2} + \varepsilon^2\left(\frac{\partial^2}{\partial r^2} - \frac{\zeta^2}{r^2}\frac{\partial^2}{\partial\zeta^2}\right) \tag{6}$$

and

$$Re_0 = R^2\Omega\rho/\mu_0 \tag{7}$$

where ρ is the density and μ_0 is the reference viscosity.

In addition, the boundary conditions on w are given by

$$w(r,0) = r^2\frac{\Omega_0}{\Omega} = r^2\kappa_0 \tag{8}$$

$$w(r,1) = r^2(1-\varepsilon^2)\frac{\Omega_1}{\Omega} = r^2(1-\varepsilon^2)\kappa_1 \tag{9}$$

and no slip requires $\partial\psi/\partial r$ and $\partial\psi/\partial\zeta$ to vanish at $\zeta = 0$ and $\zeta = 1$.

The small cone angle approximations to the solutions of equations (4)–(9), developed earlier by Turian [1], derive from an ordering scheme based on the idea of viewing the primary motion to be the purely tangential flow. Reference to equation (5) suggests that this primary flow (corresponding to the approximation $w \sim w_0$; say) must be derivable from the linear terms exclusively; inclusion of terms belonging to the nonlinear left hand member would be tantamount to the inappropriate introduction of secondary flow effects at this primary stage. Furthermore this fact and the boundary conditions in equations (8) and (9) assert that $w_0 \sim O(1)$ in the flow. Now the first correction term, say w_1, to this primary approximation must inevitably include suitable contributions from the nonlinear inertia term involving both ψ and w in equation (5), since the correction is clearly motivated by the need to compensate for their earlier exclusion. It follows then, from a comparison of the linear term involving ψ with those containing w in equation (4), that ψ is $O(Re_0\varepsilon^3)$. Accordingly we define

$$\chi = \psi/Re_0\varepsilon^3 \tag{10}$$

so that $\chi \sim O(1)$ in the flow, and equations (4) and (5) are transformed to

$$-Re^2\left[\frac{1}{r^2}\frac{\partial(\chi, E^2\chi)}{\partial(r,\zeta)} + \frac{2E^2\chi}{r^2}\left(\frac{\varepsilon^2\zeta}{1-\varepsilon^2\zeta^2}\frac{\partial\chi}{\partial r} + \frac{1}{r}\frac{\partial\chi}{\partial\zeta}\right)\right.$$
$$\left. - \frac{2}{r^2}w\left(\frac{\varepsilon^2\zeta}{1-\varepsilon^2\zeta^2}\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial\zeta}\right)\right] = E^4\chi \tag{11}$$

$$-Re^2\frac{1}{r^2}\frac{\partial(\chi, w)}{\partial(r,\zeta)} = E^2 w \tag{12}$$

in which $Re = Re_0\varepsilon^2$.

The energy equation for constant properties, written in terms of the present variables takes the form

$$-\frac{Re^2}{r^2}\frac{\partial(\chi, T)}{\partial(r,\zeta)} = \frac{1}{Pr\, r^2}\left\{\frac{\partial^2 T}{\partial\zeta^2} + \varepsilon^2\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)\right.\right.$$
$$\left.\left. - 2\zeta\frac{\partial T}{\partial\zeta}\right]\right\} + \frac{Br}{Pr}\left(\frac{\varepsilon^2\Phi_v}{\Omega^2}\right). \tag{13}$$

The dissipation function is calculated from

$$\frac{\Phi_v}{\Omega^2} = \frac{(1-\varepsilon^2\zeta^2)^2}{\varepsilon^2 r^4}\left[\frac{\partial}{\partial\zeta}\left(\frac{w}{1-\varepsilon^2\zeta^2}\right)\right]^2 + \frac{r^2}{1-\varepsilon^2\zeta^2} \times \left[\frac{\partial}{\partial r}\left(\frac{w}{r^2}\right)\right]^2 + Re^2\left\{\frac{1}{\varepsilon^2}\left[\frac{(1-\varepsilon^2\zeta^2)^{1/2}}{r^2}\chi_{\zeta\zeta} - \frac{\varepsilon^2 r}{(1-\varepsilon^2\zeta^2)^{1/2}}\frac{\partial}{\partial r}\left(\frac{\chi_r}{r^2}\right)\right]^2 + 2\left[\frac{\partial}{\partial r}\left(\frac{\chi_\zeta}{r^2}\right)\right]^2 + 2\left[\frac{\chi_\zeta}{r^3} - \frac{(1-\varepsilon^2\zeta^2)^{1/2}}{r^2}\frac{\partial}{\partial\zeta}\left(\frac{\chi_r}{(1-\varepsilon^2\zeta^2)^{1/2}}\right)\right]^2 + 2\left[\frac{\chi_\zeta}{r^3} + \frac{\varepsilon^2\zeta\chi_r}{r^2(1-\varepsilon^2\zeta^2)}\right]^2\right\}. \quad (14)$$

We develop asymptotic solutions of equation (13), for small Re and small ε, for the following two cases of thermal boundary conditions:

Case 1:

$$\bar{T}(r,0) = \bar{T}(r,1) = \bar{T}_0 \quad (15)$$

Case 2:

$$\bar{T}(r,0) = \bar{T}_0; \quad \frac{d\bar{T}}{d\zeta}(r,1) = 0 \quad (16)$$

in which the overbar denotes that the quantity involved is dimensional, say expressed in K. The dimensionless temperature appearing in equation (13) is in each case given in terms of the reference temperature $\bar{T}_0$ by $T = (\bar{T}-\bar{T}_0)/\bar{T}_0$. The thermal boundary conditions in equations (15) and (16) are identical to those used in earlier work [2] pertaining to plane Couette flow, and it was shown there that these boundary conditions correspond to the practical situation in viscometry.

ASYMPTOTIC SOLUTIONS FOR SMALL Re AND ε

The appropriate parameters governing the problem under consideration are Re^2 and ε^2 as can be deduced from equations (11), (12) and (13) and the boundary conditions (8) and (9) as well. It was shown by Turian [1] that for the hydrodynamic problem straightforward expansions for small Re for w and χ have the forms

$$w(r,\zeta;Re^2,\varepsilon^2) \sim w_0(r,\zeta;\varepsilon^2) + Re^2 w_1(r,\zeta;\varepsilon^2) + \ldots \quad (17)$$

$$\chi(r,\zeta;Re^2,\varepsilon^2) \sim \chi_0(r,\zeta;\varepsilon^2) + Re^2\chi_1(r,\zeta;\varepsilon^2) + \ldots \quad (18)$$

and further that the coefficients $w_i(r,\zeta;\varepsilon^2)$ and $\chi_i(r,\zeta;\varepsilon^2)$ have the small-ε expansions

$$w_i(r,\zeta;\varepsilon^2) \sim w_{i0}(r,\zeta) + \varepsilon^2 w_{i1}(r,\zeta) + \ldots \quad (19)$$

$$\chi_i(r,\zeta;\varepsilon^2) \sim \chi_{i0}(r,\zeta) + \varepsilon^2\chi_{i1}(r,\zeta) + \ldots. \quad (20)$$

Clearly, we are using the notation w_{ij} and X_{ij} to designate the coefficients of the terms of $O(Re^{2i}\varepsilon^{2j})$ in the expansion for w and χ, respectively.

Details of the treatment of the hydrodynamic problem embodied in equations (11) and (12) have been documented previously [1], and so have the analytical solutions for the terms w_{00}, w_{01}, w_{02}, w_{10}, X_{00}, X_{01}, and X_{10}. These solutions are developed as explicit algebraic functions of r and ζ, and can be referred to as needed. We observe that in view of the expansions in (17)–(20) the dissipation function given in equation (14) can be ordered as follows:

$$\frac{\varepsilon^2\Phi_v}{\Omega^2} = \phi_{00}(r,\zeta) + \varepsilon^2\phi_{01}(r,\zeta) + \varepsilon^4\phi_{02}(r,\zeta) + \ldots + Re^2[\phi_{10}(r,\zeta) + \varepsilon^2\phi_{11}(r,\zeta) + \ldots] + \ldots \quad (21)$$

in which, as in the foregoing, $\phi_{ij}(r,\zeta)$ is the coefficient of the term $O(Re^{2i}\varepsilon^{2j})$.

From the expansions depicted in equations (18) and (20) for χ, and (21) for Φ_v we surmise using equation (13) that the small Re^2, small ε^2 expansion for T has the form

$$T(r,\zeta;Re^2,\varepsilon^2,Pr,Br) \sim T_0(r,\zeta;\varepsilon^2,Pr,Br) + Re^2 T_1(r,\zeta;\varepsilon^2,Pr,Br) + \ldots \quad (22)$$

in which the terms T_0 and T_1 have the small ε^2 expansions given by

$$T_i(r,\zeta;\varepsilon^2,Pr,Br) \sim T_{i0}(r,\zeta;Pr,Br) + \varepsilon^2 T_{i1}(r,\zeta;Pr,Br) + \ldots. \quad (23)$$

Substituting the expansions (21)–(23) into equation (13) we obtain the following differential equations for the terms T_{ij}:

$$\frac{1}{r^2}\frac{\partial^2 T_{00}}{\partial\zeta^2} + Br\phi_{00} = 0 \quad (24)$$

$$\frac{1}{r^2}\frac{\partial^2 T_{01}}{\partial\zeta^2} + \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial T_{00}}{\partial r}\right) - 2\zeta\frac{\partial T_{00}}{\partial\zeta}\right] + Br\phi_{01} = 0 \quad (25)$$

$$\frac{1}{r^2}\frac{\partial^2 T_{02}}{\partial\zeta^2} + \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial T_{01}}{\partial r}\right) - 2\zeta\frac{\partial T_{01}}{\partial\zeta}\right] + Br\phi_{02} = 0 \quad (26)$$

$$\frac{1}{r^2 Pr}\frac{\partial^2 T_{10}}{\partial\zeta^2} + \frac{Br}{Pr}\phi_{10} = -\frac{1}{r^2}\frac{\partial(\chi_{00}, T_{00})}{\partial(r,\zeta)}. \quad (27)$$

The solutions of these equations must be carried out sequentially using the results for the hydrodynamic problem given by Turian [1]. We summarize below the solutions of equations (24)–(27) for the two cases of thermal boundary conditions given in (15) and (16):

Case 1: T(r, 0) = T(r, 1) = 0

$$T_{00} = \frac{Br}{2}(\kappa_1-\kappa_0)^2 r^2(\zeta-\zeta^2) \quad (28)$$

$$T_{01} = -\frac{Br}{12}(\kappa_1-\kappa_0)^2 r^2(\zeta^4+4\zeta^3-8\zeta^2+3\zeta) \quad (29)$$

$$T_{02} = -\frac{Br}{180}(\kappa_1-\kappa_0)^2 r^2(19\zeta^6-20\zeta^4-30\zeta^3+12\zeta^2+19\zeta) \quad (30)$$

$$T_{10} = -\frac{PrBrr^6}{50\,400}(\kappa_1-\kappa_0)^3[(\kappa_1-\kappa_0)(50\zeta^7-168\zeta^6-21\zeta^5-70\zeta^4+209\zeta)$$
$$+5\kappa_0(20\zeta^7-70\zeta^6+84\zeta^5-35\zeta^4+\zeta)]$$
$$-\frac{Brr^6}{50\,400}(\kappa_1-\kappa_0)^2[3(\kappa_1-\kappa_0)^2(25\zeta^8-98\zeta^6+56\zeta^5+126\zeta^4-112\zeta^3-18\zeta^2+21\zeta)$$
$$+2\kappa_0(\kappa_1-\kappa_0)(300\zeta^7-490\zeta^6-294\zeta^5+1260\zeta^4-980\zeta^3+175\zeta^2+29\zeta)$$
$$+140\kappa_0^2(8\zeta^6-24\zeta^5+30\zeta^4-20\zeta^3+7\zeta^2-\zeta)]. \quad (31)$$

Case 2: $T(r, 0) = 0$, $\frac{\partial T}{\partial \zeta}(r, 1) = 0$

$$T_{00} = \frac{Br}{2}(\kappa_1-\kappa_0)^2 r^2(2\zeta-\zeta^2) \quad (32)$$

$$T_{01} = -\frac{Br}{12}(\kappa_1-\kappa_0)^2 r^2(\zeta^4+8\zeta^3-8\zeta^2-12\zeta) \quad (33)$$

$$T_{02} = -\frac{Br}{180}(\kappa_1-\kappa_0)^2 r^2(19\zeta^6-20\zeta^4+120\zeta^3-394\zeta) \quad (34)$$

$$T_{10} = -\frac{PrBrr^6}{25\,200}(\kappa_1-\kappa_0)^3[(\kappa_1-\kappa_0)(50\zeta^7-84\zeta^6+63\zeta^5-70\zeta^4+119\zeta)$$
$$+5\kappa_0(10\zeta^7-14\zeta^6+21\zeta^5-35\zeta^4+49\zeta)]$$
$$-\frac{Brr^6}{50\,400}(\kappa_1-\kappa_0)^2[3(\kappa_1-\kappa_0)^2(25\zeta^8-98\zeta^6+56\zeta^5+126\zeta^4-112\zeta^3-18\zeta^2-24\zeta)$$
$$+2\kappa_0(\kappa_1-\kappa_0)(300\zeta^7-490\zeta^6-294\zeta^5+1260\zeta^4-980\zeta^3+175\zeta^2-140\zeta)$$
$$+140\kappa_0^2(8\zeta^6-24\zeta^5+30\zeta^4-20\zeta^3+7\zeta^2-2\zeta)]. \quad (35)$$

In these equations $\kappa_0 = \Omega_0/\Omega$, and $\kappa_1 = \Omega_1/\Omega$ as implied in equations (8) and (9).

It should be observed that the first term in the approximation, T_{00}, in each case, corresponds to the plane Couette flow situation [equations (28) and (32)]. This will be exploited later by utilizing available results for this simpler geometry to assess the influence of the temperature sensitivity of the transport properties on the heat-transfer problem under consideration here.

CALCULATION OF T_{max}, Q, AND Nu

The maximum temperature $T_{max} = T(1, \zeta_m)$ occurs at $\zeta = \zeta_m$, which is determined from a solution of

$$\frac{\partial T}{\partial \zeta}(1, \zeta_m) = 0. \quad (36)$$

Explicit analytical expansions for ζ_m can be developed through enforcement of the condition in (36) and use of a suitable process of inversion. To facilitate this it is convenient to utilize the general form

$$\zeta_m = \zeta_{00}[1+\varepsilon^2\zeta_{01}+\ldots+Re^2(\zeta_{10}+\ldots)+\ldots]. \quad (37)$$

When condition (36) is imposed on the temperature distribution for Case 1 given by equations (28)–(31), and equation (37) is used to enforce the condition at successive levels of approximation, one gets

$$\zeta_m = \frac{1}{2}\left\{1+\frac{\varepsilon^2}{4}+\frac{\varepsilon^4}{480}+\ldots-Re^2\left[Pr\left(\frac{181(\kappa_1-\kappa_0)^2}{32\,256}-\frac{19\kappa_0(\kappa_1-\kappa_0)}{80\,640}-\frac{277(\kappa_1-\kappa_0)(\kappa_1+\kappa_0)}{134\,400}\right]+\ldots\right\}. \quad (38)$$

Also T_{max} for Case 1 is calculated from equations (28)–(31) by using equation (38) in $T_{max} = T(1, \zeta m)$ and enforcing the asymptotic ordering scheme. This gives

$$T_{max} = \frac{Br}{8}(\kappa_1-\kappa_0)^2\left(1-\frac{\varepsilon^2}{24}-\frac{409}{1440}\varepsilon^4-\ldots\right)-\frac{7Re^2Br(\kappa_1-\kappa_0)^4}{1\,843\,200}(508Pr+3)+\ldots. \quad (39)$$

For Case 2 it is obvious from equation (16) that $T_{max} = T(1, 1)$ and equations (32)–(35) then yield

$$T_{max} = \frac{Br}{2}(\kappa_1-\kappa_0)^2\left(1+\frac{11}{6}\varepsilon^2+\frac{55}{18}\varepsilon^4+\ldots\right)$$
$$-\frac{Re^2Br(\kappa_1-\kappa_0)^2}{50\,400}[3(\kappa_1-\kappa_0)^2(52Pr-45)+2\kappa_0(\kappa_1-\kappa_0)(155Pr-169)-140\kappa_0^2]+\ldots. \quad (40)$$

Next we calculate the total rate of heat generation $\bar{Q} = 2\pi R^3 \varepsilon \rho C_p \bar{T}_0 Q$, in which the nondimensional heat generation rate Q is calculated from

$$Q = \int_0^1 \int_0^1 T(r, \zeta) r^2 \, dr \, d\zeta. \tag{41}$$

In addition the Nusselt number is calculated using the convenient definition

$$Nu(r, \zeta) = \frac{hR\varepsilon}{k_0} = \frac{(1-\varepsilon^2\zeta^2)^{1/2}}{r} \frac{\partial T}{\partial \zeta} \tag{42}$$

where h is the local heat-transfer coefficient. Combining the solutions for the temperature field, equations (28)–(31) for Case 1, and (32)–(35) for Case 2, with equations (41) and (42) we get the following results:

Case 1: $T(r, 0) = T(r, 1) = 0$

$$Q = \frac{Br}{60}(\kappa_1-\kappa_0)^2\left(1-\frac{\varepsilon^2}{30}-\frac{11\varepsilon^4}{35}-\ldots\right)-\frac{277Re^2PrBr(\kappa_1-\kappa_0)^4}{1\,814\,400} + \frac{Re^2Br(\kappa_1-\kappa_0)^2}{13\,608\,000}[17(\kappa_1-\kappa_0)^2+100\kappa_0\kappa_1]+\ldots \tag{43}$$

$$Nu(r, 0) = \frac{Br}{2}(\kappa_1-\kappa_0)^2 r\left(1-\frac{\varepsilon^2}{2}-\frac{19}{180}\varepsilon^4-\ldots\right)-\frac{Re^2PrBrr^5}{50\,400}(\kappa_1-\kappa_0)^3(209\kappa_1-204\kappa_0) - \frac{Re^2Brr^5}{50\,400}(\kappa_1-\kappa_0)^2[63(\kappa_1-\kappa_0)^2+58\kappa_0(\kappa_1-\kappa_0)-140\kappa_0^2]+\ldots \tag{44}$$

$$Nu(r, 1) = -\frac{Br}{2}(\kappa_1-\kappa_0)^2 r\left|1-\varepsilon^2(0)-\frac{161}{360}\varepsilon^4-\ldots\right|+\frac{Re^2PrBrr^5}{50\,400}(\kappa_1-\kappa_0)^3(834\kappa_1-839\kappa_0) - \frac{Re^2Brr^5}{50\,400}(\kappa_1-\kappa_0)^2[135(\kappa_1-\kappa_0)^2+338\kappa_0(\kappa_1-\kappa_0)+140\kappa_0^2]+\ldots. \tag{45}$$

Case 2: $T(r, 0) = 0, \; \frac{\partial T}{\partial \zeta}(r, 1) = 0$

$$Q = \frac{Br}{15}(\kappa_1-\kappa_0)^2\left(1+\frac{97}{60}\varepsilon^2+\frac{589}{210}\varepsilon^4+\ldots\right)-\frac{Re^2PrBr(\kappa_1-\kappa_0)^3}{302\,400}(67\kappa_1+68\kappa_0) + \frac{Re^2Br(\kappa_1-\kappa_0)^2}{6\,804\,000}[1021(\kappa_1-\kappa_0)^2+2585\kappa_0(\kappa_1-\kappa_0)+1100\kappa_0^2]+\ldots \tag{46}$$

$$Nu(r, 0) = Br(\kappa_1-\kappa_0)r\left(1+\varepsilon^2+\frac{197}{90}\varepsilon^4+\ldots\right)-\frac{Re^2PrBrr^5(\kappa_1-\kappa_0)^3}{3600}(17\kappa_1+18\kappa_0) + \frac{Re^2Brr^5(\kappa_1-\kappa_0)^2}{6300}[9(\kappa_1-\kappa_0)^2+35\kappa_0\kappa_1]+\ldots. \tag{47}$$

For Case 2 the local Nusselt number over the cone surface $Nu(r, 1) = 0$ of course.

It is useful to reconsider the relationship between the differential equations (24)–(27) and the foregoing results obtained from them. Equation (24) contains a conduction term and a frictional heating source term. It represents precisely the plane Couette flow approximation to the motion considered, and this therefore is the reference primary approximation. Equations (25) and (26) represent corrections necessitated by the differing geometry, and these can be made as small as desired by taking ε sufficiently small. Equation (27) contains a conductive, a source and a convective contribution. The viscous heating contribution from this equation appears in the terms in (Re^2Br) in equations (39), (40) and (43)–(47), whereas the convective contribution is contained in the terms in (Re^2PrBr). It is clear that in the limit $\varepsilon^2 \to 0$, $Re^2 \to 0$ the plane Couette flow results are recovered, and this common root will now be examined further.

PLANE COUETTE FLOW OF VARIABLE PROPERTY FLUID

The frictional heating in plane Couette flow of a fluid with variable transport properties was treated by Turian and Bird [2] before. Among the various situations examined, it was shown that for a fluid with transport properties obeying the relationships

$$\frac{k}{k_0} = 1+\alpha_1 T+\alpha_2 T^2+\ldots \tag{48}$$

$$\frac{\mu_0}{\mu} = e^{\beta T} = 1+\beta_1 T+\beta_2 T^2+\ldots \tag{49}$$

the approximations for the temperature field for small Br, when adapted to the cone-and-plate geometry, yield the following:

Case 1:

$$T = \frac{Br}{2}(\kappa_1-\kappa_0)^2 r^2(\zeta-\zeta^2) + \frac{Br^2\beta_1}{24}(\kappa_1-\kappa_0)^3 r^4[(\kappa_1-\kappa_0)(\zeta^4-2\zeta^3+\zeta)+2(\zeta^2-\zeta)] - \frac{Br^2\alpha_1}{8}(\kappa_1-\kappa_0)^4 r^4(\zeta^2-\zeta)^2 + \dots \quad (50)$$

Case 2:

$$T = \frac{Br}{2}(\kappa_1-\kappa_0)^2 r^2(2\zeta-\zeta^2) + \frac{Br^2\beta_1}{24}(\kappa_1-\kappa_0)^3 r^4[(\kappa_1-\kappa_0)(\zeta^4-4\zeta^3+8\zeta)+8(\zeta^2-\zeta)] - \frac{Br^2\alpha_1}{8}(\kappa_1-\kappa_0)^4 r^4(\zeta^2-2\zeta)^2 + \dots \quad (51)$$

As indicated by Turian and Bird [2] the process of adapting the plane Couette flow results to the cone and plate results given by equations (50) and (51) above, and also those given by equations (52)–(59) below, entails making the following simple replacements:

Plane Couette flow	Cone and plate flow
x/b	$(r\sin\beta/r\sin\beta_0) = \zeta$
$Br_c = \dfrac{\mu_0 V^2}{k_0 T_0}$	$\dfrac{\mu_0 r^2\Omega^2}{k_0 T_0} = Br(r^2)$

where b is the channel separation and the reference speed $V = |V_0| + |V_1|$, with V_0 and V_1 being the velocities of the planes at $x = 0$ and $x = b$, respectively.

Equations (50) and (51) yield the following results for ζ_m, T_{max}, Q and Nu calculated in the same way as before:

Case 1: $T(r, 0) = T(r, 1) = 0$

$$\zeta_m = \tfrac{1}{2}$$

$$T_{max} = \frac{Br}{8}(\kappa_1-\kappa_0)^2 + Br^2\beta_1(\kappa_1-\kappa_0)^3\left[\frac{5}{384}(\kappa_1-\kappa_0)-\frac{1}{48}\right] - \frac{Br^2\alpha_1}{128}(\kappa_1-\kappa_0)^4 + \dots \quad (52)$$

$$Q = \frac{Br}{60}(\kappa_1-\kappa_0)^2 + \frac{Br^2\beta_1}{2520}(\kappa_1-\kappa_0)^3[3(\kappa_1-\kappa_0)-5] - \frac{Br^2\alpha_1}{1680}(\kappa_1-\kappa_0)^4 + \dots \quad (53)$$

$$Nu(r, 0) = \frac{Br}{2}(\kappa_1-\kappa_0)r + \frac{Br^2\beta_1}{24}(\kappa_1-\kappa_0)^3 r^3[(\kappa_1-\kappa_0)-2] + Br^2\alpha_1(0) + \dots \quad (54)$$

$$Nu(r, 1) = -\frac{Br}{2}(\kappa_1-\kappa_0)r - \frac{Br^2\beta_1}{24}(\kappa_1-\kappa_0)^3 r^3[(\kappa_1-\kappa_0)-2] - Br^2\alpha_1(0) + \dots \quad (55)$$

Case 2: $T(r, 0) = 0,\ \dfrac{\partial T}{\partial \zeta}(r, 1) = 0$

$$\zeta_m = 1$$

$$T_{max} = \frac{Br}{2}(\kappa_1-\kappa_0)^2 + \frac{Br^2\beta_1}{24}(\kappa_1-\kappa_0)^3[5(\kappa_1-\kappa_0)-8] - \frac{Br^2\alpha_1}{8}(\kappa_1-\kappa_0)^4 + \dots \quad (56)$$

$$Q = \frac{Br}{15}(\kappa_1-\kappa_0)^2 + \frac{2Br^2\beta_1}{315}(\kappa_1-\kappa_0)^3[3(\kappa_1-\kappa_0)-5] - \frac{Br^2\alpha_1}{105}(\kappa_1-\kappa_0)^4 + \dots \quad (57)$$

$$Nu(r, 0) = Br(\kappa_1-\kappa_0)r + \frac{Br^2\beta_1}{3}(\kappa_1-\kappa_0)^3 r^3[(\kappa_1-\kappa_0)-2] + Br^2\alpha_1(0) + \dots \quad (58)$$

$$Nu(r, 1) = 0. \quad (59)$$

It is clear that all of the correction terms in the equations in this section, i.e. the terms containing α_1 and β_1 in equations (50)–(59), account for the temperature sensitivity of the thermal conductivity and the viscosity. Thus these are small Br expansions pertaining to situations in which only small temperature gradients are generated within the fluid, and the primary reference state is for a fluid with constant transport properties coinciding with those evaluated at the reference temperature $\bar{T}_0$.

The asymptotic results in the preceding sections, equations (28)–(47), were developed with Br taken as a free parameter, and they might appear to be valid for arbitrary values of Br, except for the fact that large Br values cannot be tolerated since they give rise to large temperature gradients, and therefore to the collapse of the underlying assumption that the transport properties are fixed. It follows then that the basic tie between the results of the preceding sections and those quoted in this section resides in the following: the small Re^2, small ε^2 results of the preceding sections are also implicitly restricted to small Br values consistent with the explicit assumption that the transport properties are constant, whereas the small Br results of this section are also restricted to small Re^2 and ε^2 values in order to sustain the assumption that geometric and secondary flow effects are small (which clearly are implicitly made in adapting the plane Couette flow results to cone and plate flow).

INTERPRETATION OF RESULTS

It is instructive to examine the calculations presented in the preceding sections in the context of pertinent previously available results. It will be convenient, without being limiting, to consider for this purpose only the situation for which the plate is held fixed while the cone is rotated at a uniform speed, i.e. for $\kappa_0 = 0$ and $\kappa_1 = 1$. Under these circumstances equations (39) and (52) for $T_{\max}$ in Case 1 reduce to

$$T_{\max} = \frac{Br}{8}\left\{1 - \frac{\varepsilon^2}{24} - \frac{409}{1440}\varepsilon^4 - \dots \right.$$
$$\left. \dots - \frac{889 Re^2 Pr}{230\,400} - \frac{7Re^2}{76\,800} + \dots \right\} \quad (60)$$

and

$$T_{\max} = \frac{Br}{8}\left\{1 - \frac{Br\beta_1}{16} - \frac{Br\alpha_1}{16} + \dots \right\} \quad (61)$$

while equations (43) and (53) for Q for the same case yield

$$Q = \frac{Br}{60}\left\{1 - \frac{\varepsilon^2}{30} - \frac{11}{35}\varepsilon^4 - \dots \right.$$
$$\left. \dots - \frac{277 Re^2 Pr}{30\,240} + \frac{17Re^2}{226\,800} + \dots \right\} \quad (62)$$

and

$$Q = \frac{Br}{60}\left\{1 + \frac{Br\beta_1}{14} - \frac{Br\alpha_1}{28} + \dots \right\}. \quad (63)$$

The expressions for $T_{\max}$ for Case 2 are obtained from equations (40) and (56), and they reduce to the forms

$$T_{\max} = \frac{Br}{2}\left\{1 + \frac{11}{6}\varepsilon^2 + \frac{55}{18}\varepsilon^4 + \dots \right.$$
$$\left. \dots - \frac{13 Re^2 Pr}{2100} - \frac{3Re^2}{560} + \dots \right\} \quad (64)$$

and

$$T_{\max} = \frac{Br}{2}\left\{1 - \frac{Br\beta_1}{4} - \frac{Br\alpha_1}{4} + \dots \right\}. \quad (65)$$

For Case 2 the reduced forms of the expressions for Q, obtained from equations (46) and (57), are given by

$$Q = \frac{Br}{15}\left\{1 + \frac{97}{60}\varepsilon^2 + \frac{589}{210}\varepsilon^4 + \dots \right.$$
$$\left. \dots - \frac{67 Re^2 Pr}{20\,160} + \frac{1021 Re^2}{453\,600} + \dots \right\} \quad (66)$$

and

$$Q = \frac{Br}{15}\left\{1 + \frac{2}{7}Br\beta_1 - \frac{Br\alpha_1}{7} + \dots \right\}. \quad (67)$$

It is useful to recall now certain previously established results to ascertain whether they impose possible restrictions on the present expansions, and beyond that to explore their utility as guidelines for assessing the ranges of the present approximations. First, for strictly the case of plane Couette flow it has been established (see Joseph [3, 4]) that value of $(B\bar{\beta})$ corresponding to the critical stress is equal to 18.3 and 4.6 for Cases 1 and 2, respectively. These values turn out to be considerably in excess of the maximum $(Br\beta)$ values for which the expansions (61), (63), (65) and (67) retain any meaning as asymptotic approximations. Accordingly these expansions are restricted to the lower maximum temperature branch of the critical stress-maximum temperature rise curve characteristic of this type of flow [3, 4, 6], although the temperature and velocity distributions are strictly single-valued when expressed in terms of the Brinkman number.

The torque on the cone required to maintain the motion will naturally be affected by the temperature variation of the transport properties, and also by the intensity of the secondary motions. A pertinent result deriving from the plane Couette flow analysis is the ratio (M/M_0), where M is the actual torque, and M_0 is the torque value corresponding to the situation when the fluid transport properties are assumed to be unaffected by the temperature gradients. It was shown previously [2] that this ratio is given by

$$(M/M_0) = 1 - \frac{Br\beta_1}{20} + \dots \quad (68)$$

for Case 1, and by

$$(M/M_0) = 1 - \frac{Br\beta_1}{5} + \dots \quad (69)$$

for Case 2.

It is interesting that to the first order in Br the temperature dependence of the thermal conductivity does not affect the torque.

In the analysis of the three-dimensional flow in the cone and plate system by Turian [1] a different sort of torque ratio, $(\tilde{M}/M_0)$, from those given by equations (68) and (69) was calculated. In this instance the torque $\tilde{M}$ is the value required to sustain the actual three-dimensional flow patterns, while M_0 is again the torque value due merely to the one-dimensional, purely tangential, primary motion assumed unaffected by the secondary flows. The torque ratio so defined is found to be given by

$$(\tilde{M}/M_0) = 1 + \frac{3}{4900} Re^2 + \ldots . \qquad (70)$$

Using a rather extensive collection of available experimental data it was discovered that the simple result in equation (70) has a surprisingly broad range [1], extending virtually to the limit when the correction term $(3Re^3/4900)$ is equal to the first term 1. But the other approximations above are more restricted as will be shown below.

It should be observed that for $(Br\beta_1)$ and also (Re^2) small the effects of temperature rise and secondary flows on the measured torque can be estimated by linear superposition of the correction terms in equations (68)–(70). The more frequent situations, however, turn out to be those in which these effects are mutually exclusive; for highly viscous, temperature-sensitive, liquids the effects of temperature gradients on the viscosity predominate (usually exclusively) those of the secondary motions, while for highly fluid liquids (which also happen to be relatively temperature insensitive) the reverse situation applies. This is demonstrated in Table 1 in which the terms in equations (60)–(67) for a typical lubricating oil (Schlichting [14]) and for water are given. The reference temperature used in 293 K for both liquids. The column entries represent the absolute values of the terms within the brackets in equations (60)–(67), except for the first column with the heading $\overline{(Br)}$, which gives the values of the term outside the brackets in each case. Thus for Case 1 the first two rows in the column headed by $\overline{(Re^2Pr)}$ gives the values of the term $(889Re^2Pr/230\,400)$ in equation (60) for the lubricating oil and for water, while the first two rows in the $\overline{(Br)}$ column give the values of the term $(Br/8)$ in equation (60). The columns headed $\overline{(Br\beta_1)}$ and $\overline{(Br\alpha_1)}$ give the values of $(Br\beta_1/16)$ and $(Br\alpha_1/16)$, and so on. It is clear from this table that for the lubricating oil the term $(Br\beta_1)$ represents the most significant effect even though the value of Pr for this oil is 10 100. For water the inertial effects embodied in the terms (Re^2Pr) are clearly most important, although even for Case 2 the value $(Br/2) = 2.8 \times 10^{-6}$ corresponds to a maximum temperature rise of only 0.0008 K. For the lubricating oil, however, the value $(Br/2) = 5.84 \times 10^{-4}$ corresponds to a maximum temperature rise of 0.17 K, and from equation (69) the corresponding correction, $(Br\beta_1/5)$, to the measured torque is equal to 0.004, while the term $(3Re^2/4900)$ which represents the correction due to the secondary motions from equation (70) is only equal to 1.12×10^{-8}. The temperature distributions within the fluid gap are depicted in Fig. 2 for water, and in Fig. 3 for the lubricating oil for selected operating conditions for Case 1. Comparison of the curves for the case when secondary flow effects are included, equation (60), with those corresponding to the plane Couette flow approximation, equation (61), shows that one additional important consequence of the inclusion of the convective terms is the loss of symmetry about the "mid-plane" $\zeta = 1/2$.

In order to utilize the results obtained in the foregoing it is necessary to establish first the nature of the restrictions appropriate to the situation considered. As a general guide it will be sufficient to examine the various restrictions intrinsic to and imposed by each of the applicable approximations in any given situation and to identify the approximation which is most restrictive, which would then serve as the

Table 1. Comparison of expansion terms for viscous oil and water

Expansion term of order:		$\overline{(Br)}$	$\overline{(\varepsilon^2)}$	$\overline{(Re^2Pr)}$	$\overline{(Re^2)}$	$\overline{(Br\beta_1)}$	$\overline{(Br\alpha_1)}$
		Case 1: $T(r, 0) = T(r, 1) = 0$					
$T_{max}[(60), (61)]$:	Lubr. oil	1.46×10^{-4}	1.27×10^{-5}	7.10×10^{-4}	$1.6 \times 10^{\ 9}$	1.29×10^{-3}	1.72×10^{-5}
	Water	7.00×10^{-7}	7.93×10^{-7}	9.72×10^{-2}	8.16×10^{-5}	1.94×10^{-6}	2.23×10^{-7}
$Q[(62), (63)]$:	L br. oil	1.95×10^{-5}	1.02×10^{-5}	1.69×10^{-3}	1.3×10^{-9}	1.47×10^{-3}	9.84×10^{-6}
	Water	9.34×10^{-8}	6.35×10^{-7}	5.77×10^{-2}	6.71×10^{-5}	2.21×10^{-6}	1.27×10^{-7}
		Case 2: $T(r, 0) = [\partial T(r, 1)/\partial\zeta] = 0$					
$T_{max}[(64), (65)]$:	Lubr. oil	5.84×10^{-4}	5.58×10^{-4}	1.14×10^{-3}	9.75×10^{-8}	5.16×10^{-3}	6.89×10^{-5}
	Water	2.80×10^{-6}	3.49×10^{-5}	3.90×10^{-2}	4.80×10^{-3}	7.75×10^{-6}	8.91×10^{-7}
$Q[(66), (67)]$:	Lubr. oil	7.79×10^{-5}	4.92×10^{-4}	6.12×10^{-4}	4.10×10^{-8}	5.91×10^{-3}	3.95×10^{-4}
	Water	3.74×10^{-7}	3.08×10^{-5}	2.09×10^{-2}	2.02×10^{-3}	8.85×10^{-6}	5.09×10^{-7}

	R (cm)	ε	Ω (rad/s)	μ_0 (P)	β_1	α_1	Re	Br	Pr
Lubr. oil	5.0	sin 1°	5	7.96	17.67	−0.236	4.27×10^{-3}	1.17×10^{-3}	10 100
Water	5.0	sin $\frac{1}{4}$°	20	0.01	5.53	0.636	9.46×10^{-1}	5.60×10^{-6}	7.03

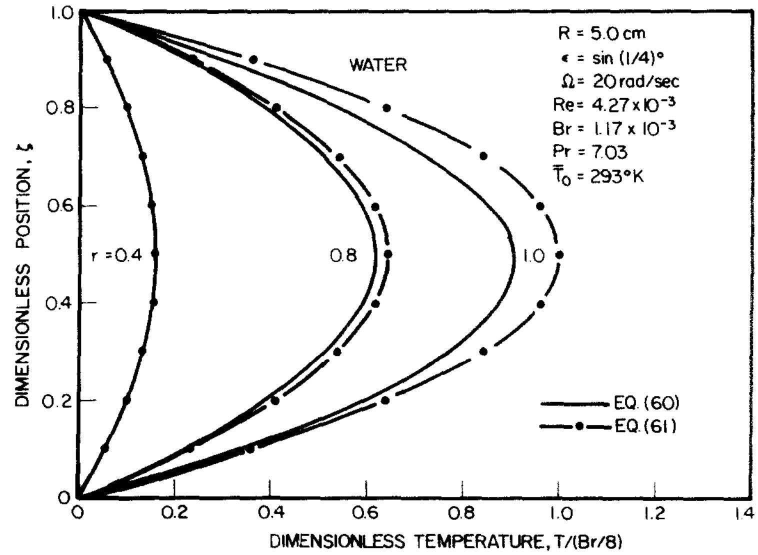

FIG. 2. Temperature distributions for water.

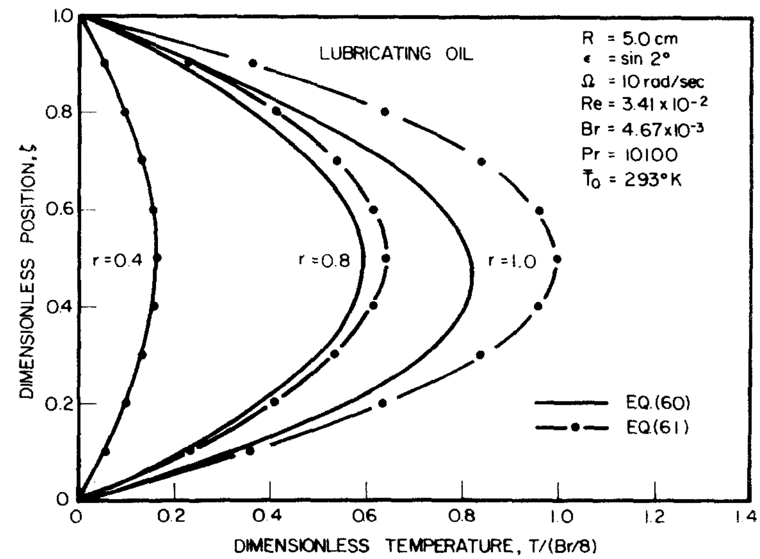

FIG. 3. Temperature distributions for lubricating oil.

governing restriction. For example, for the thermal boundary conditions corresponding to Case 1, the applicable equations are: for T_{max} equations (60) and (61), for Q equations (62) and (63), and for the torque ratios equations (68) and (70). As asymptotic relations we might arbitrarily invoke the rule that the dominant correction term in each relation cannot exceed a certain fraction (say 10%) of the primary fundamental approximation term. For the relations in equations (60)–(63), for instance, the dominant correction terms are, for liquids, usually the terms $(889Re^2Pr/230\,400)$, $(Br\beta_1/16)$, $(277Re^2Pr/30\,240)$, and $(Br\beta_1/14)$, respectively. The pertinent correction terms for the torque relations in equations (68) and (70) are clearly $(Br\beta_1/20)$ and $(3Re^2/4900)$. It is a trivial matter to establish which of these six correction terms embodies the most stringent restriction in any given situation.

CONCLUSIONS

The results presented here, taken together with related earlier studies [1, 2], provide a fairly complete analytical treatment of the hydrodynamic and frictional heating problem in the small angle cone and plate system. The small angle property characteristic of most practical cone and plate systems is clearly the key to the effectiveness of the asymptotic scheme in establishing results of rather broad applicability. This body of work was motivated by the need to establish quantitative guidelines for cone and plate viscometry, but the analysis can also be used to pattern mass transfer calculations in this system. It would seem that this geometry should prove useful in mass transfer and related studies in view of the well defined hydrodynamic problem, and the availability of simple analytical results relating to it.

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CONVECTION ET CHAUFFAGE AERODYNAMIQUE POUR UN SYSTEME DE CONE ET DE PLAN

Résumé—On calcule le champ de température résultant du chauffage aérodynamique d'un fluide confiné dans la région entre un cone et un plan, lesquels sont en rotation relative, les conditions aux limites thermiques étant données. Le problème de convection relatif à l'écoulement tridimensionnel, caractéristique de cette géométrie, est résolu sous la forme d'un développement asymptotique pour un petit nombre de Reynolds et un petit angle. Avec l'adaptation de résultats précédemment publiés sur l'écoulement plan, de Couette de fluides à propriétés de transport dépendant de la température, ces solutions fournissent l'estimation des effets de la conduction, de la convection, du chauffage aérodynamique aussi bien que de la sensibilité des propriétés de transport à la température.

Zusammenfassung—Das Temperaturfeld, das durch dissipative Erwärmung eines Fluids im Bereich zwischen einem rotierenden Kegel und einer Platte entsteht, wird für vorgegebene thermische Grenzbedingungen berechnet. Das konvektive Wärmeübertragungsproblem, das zu dem dreidimensionalen Geschwindigkeitsfeld gehört, wird als eine asymptotische Erweiterung für kleine Reynoldszahlen und Kegel–Winkel gelöst. Zusammen mit Anpassungen aus früher publizierten Ergebnissen für ebene Couette–Strömung von Fluiden mit temperaturabhängigen Transporteigenschaften liefern diese Lösungen ein quantitatives Maß für die Effekte der Leitung, Konvektion, Dissipation ebenso wie für die Temperaturabhängigkeit der Transporteigenschaften des Fluids.

КОНВЕКЦИЯ И ДИССИПАТИВНЫЙ НАГРЕВ В СИСТЕМЕ КОНУС-ПЛАСТИНА

Аннотация — Для заданных граничных условий дается расчет температурного поля, обусловленного диссипативным нагревом жидкости в зазоре между конусом и пластиной при их стационарном вращении относительно друг друга. Задача конвективного переноса тепла в характерном для данной геометрии трехмерном поле скоростей решена с помощью асимптотического разложения в ряд для малых чисел Рейнольдса и малого угла конусности. Вместе с ранее опубликованными результатами для плоского течения Куэтта жидкостей с зависящими от температуры свойствами, эти решения дают количественные оценки эффектов теплопроводности, конвекции, диссипативного нагрева, а также температурной зависимости переносных свойств жидкости.